Math 301A Problem Set #5 – Solution 1 October 2008

1. (Based on Chapter 5, Exercise 18) Let

$$\alpha = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 1 & 3 & 5 & 4 & 7 & 6 & 8 \end{bmatrix} \quad \text{and} \quad \beta = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 3 & 8 & 7 & 6 & 5 & 2 & 4 \end{bmatrix}$$

(a) Write α, β, α^{-1}, and β^{-1} as products of disjoint cycles.

Solution: We have

$$\begin{aligned} \alpha = (1\,2)(4\,5)(6\,7) \quad &\text{and} \quad \beta = (2\,3\,8\,4\,7)(5\,6); \\ \alpha^{-1} = \alpha \quad &\text{and} \quad \beta^{-1} = (2\,7\,4\,8\,3)(5\,6) \end{aligned}$$

(b) Write α, β, α^{-1}, and β^{-1} as products of 2-cycles.

Solution: The elements α and α^{-1} are already written as 2-cycles above. Following the proof of Theorem 5.4, we write β as a 2-cycle. We get

$$\beta \;=\; (2\,7)(2\,4)(2\,8)(2\,3)(5\,6)$$

Now we can invert this just by writing the transpositions in the opposite order:

$$\beta^{-1} = (5\,6)(2\,3)(2\,8)(2\,4)(2\,7)$$

2. (Based on Chapter 5, exercise 28) Let $\beta = (1\ 2\ 3)(1\ 4\ 5)$. Write β^{99} in cycle form. Let $\gamma = (1\ 2\ 3\ 4)(2\ 4)(3\ 5)$. Write γ^{99} in cycle form.

Solution: First we rewrite β as a product of disjoint cycles. We get

$$\beta \;=\; (1\ 4\ 5\ 2\ 3).$$

Now since β is a 5-cycle, it has order 5, and we get

$$\beta^{99} = \beta^{95}\beta^{4} = (\beta^{5})^{19}\beta^{4} = \beta^{4} = \beta^{-1}.$$

To find an expression for β^{-1}, we just write the cycle above in reverse order:

$$\beta^{99} = \beta^{-1} = (1\ 3\ 2\ 5\ 4).$$

We rewrite γ as a product of disjoint cycles. We get

$$\gamma \;=\; (1\ 2)(3\ 5\ 4).$$

By Theorem 5.3, we know the order of γ is 6, so we write

$$\gamma^{99} = \gamma^{96}\gamma^{3} = (\gamma^{6})^{16}\gamma^{3} = \gamma^{3}.$$

Since the two cycles in γ commute, we can raise them to the power 3 independently. The 3-cycle has order 3, so its third power is the identity. The 2-cycle has order 2, so its third power is itself. We get

$$\gamma^{99} = \gamma^{3} = (1\ 2).$$

3. (Based on Chapter 5, Exercises 8 and 24) In the group S_7,

(a) How many elements have order 5?

Solution: An element of S_7 has order 5 if and only if it is a 5-cycle. We can count the number of 5-cycles as follows: The first entry can be any of the 7 numbers in $\{1, 2, 3, 4, 5, 6, 7\}$. The second entry can be any of the remaining 6; the third can be any of the remaining 5, and so on. We get

$$7 \times 6 \times 5 \times 4 \times 3 = 2520$$

expressions of the form $(a_1\, a_2\, a_3\, a_4\, a_5)$. However, they are not all distinct elements of S_7. We have counted each 5-cycle five times, because, for instance, $(1\,2\,3\,4\,5)$ is the same as $(2\,3\,4\,5\,1)$. So the actual number of 5-cycles in S_7 is $2520/5 = 504$.

(b) What is the maximum order of any element?

Solution: We can write every element of S_7 as a product of disjoint cycles whose lengths (when we include the 1-cycles) add up to 7. A generalization of Theorem 5.3 then tells us that the order of an product of disjoint cycles of lengths $l_1, l_2, \ldots, l_k$ is the least common multiple of $l_1, l_2, \ldots, l_k$.

To determine the maximum order of any element of S_7, we consider all the possible ways to write 7 as a sum of positive integers, and use Theorem 5.3 to find the order of a product of cycles of the corresponding lengths.

Sum	Cycle structure	Order
1+1+1+1+1+1+1	$(\cdot)$	1
1+1+1+1+1+2	$(\cdot\,\cdot)$	2
1+1+1+1+3	$(\cdot\;\cdot\;\cdot)$	3
1+1+1+2+2	$(\cdot\,\cdot)(\cdot\,\cdot)$	2
1+1+1+4	$(\cdot\;\cdot\;\cdot\;\cdot)$	4
1+1+2+3	$(\cdot\,\cdot)(\cdot\;\cdot\;\cdot)$	6
1+2+2+2	$(\cdot\,\cdot)(\cdot\,\cdot)(\cdot\,\cdot)$	2
1+1+5	$(\cdot\;\cdot\;\cdot\;\cdot\;\cdot)$	5
1+2+4	$(\cdot\,\cdot)(\cdot\;\cdot\;\cdot\;\cdot)$	4
1+3+3	$(\cdot\;\cdot\;\cdot)(\cdot\;\cdot\;\cdot)$	3
2+2+3	$(\cdot\,\cdot)(\cdot\,\cdot)(\cdot\;\cdot\;\cdot)$	6
1+6	$(\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot)$	6
2+5	$(\cdot\,\cdot)(\cdot\;\cdot\;\cdot\;\cdot\;\cdot)$	10
3+4	$(\cdot\;\cdot\;\cdot)(\cdot\;\cdot\;\cdot\;\cdot)$	12
7	$(\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot)$	7

It appears that the maximum order is 12 for a disjoint product of a 3-cycle and a 4-cycle.

(c) How many elements have order 6?

Solution: Elements of order 6 come in three possible cycle structures:

$$(\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot), \quad (\cdot\;\cdot\;\cdot)(\cdot\,\cdot), \quad \text{and } (\cdot\;\cdot\;\cdot)(\cdot\,\cdot)(\cdot\,\cdot).$$

The number of 6-cycles (following the logic in part (a)) is

$$\frac{7 \times 6 \times 5 \times 4 \times 3 \times 2}{6} = 840.$$

To find the number of elements with cycle structure $(\cdot\;\cdot\;\cdot)(\cdot\,\cdot)$, we start by filling the 3-cycle. There are $\frac{7 \times 6 \times 5}{3} = 70$ ways to do this. From the remaining 4 letters, we choose two for the two-cycle. There are $\binom{4}{2} = 6$ ways to do this. Altogether, then, we get

$$70 \times 6 = 420$$

elements with cycle structure $(\cdot\;\cdot\;\cdot)(\cdot\,\cdot)$.

To find the number of elements with cycle structure $(\cdot\ \cdot\ \cdot)(\cdot\,\cdot)(\cdot\,\cdot)$, we start, as before, by filling the 3-cycle. There are $\dfrac{7 \times 6 \times 5}{3} = 70$ ways to do this. Now we need the number of ways to partition the remaining four letters into two pairs. It turns out that there are three ways to do this. Here's the argument: let a denote the least of the remaining four letters. We choose another letter to be in a 2-cycle with a. There are three ways to do this. Then the remaining two letters form the other 2-cycle; there's no more freedom to choose. Thus we have

$$70 \times 3 \quad = \quad 210$$

elements with cycle structure $(\cdot\ \cdot\ \cdot)(\cdot\,\cdot)(\cdot\,\cdot)$.

Summing up, we get $840 + 420 + 210 = 1470$ elements of order 6 in S_7.