MAT 428 Review Sheet 1

The questions on Exam 1 will be similar to questions here.

1) Define each of the following:
   a) Archimedean property
   b) bounded sequence
   c) countable set
   d) bounded set
   e) convergence of a sequence
   f) completeness axiom
   g) the set of real numbers
   h) density of the rationals
   i) algebraic number
2) Give examples of each of the following:
   a) A sequence that is bounded but not monotone.
   b) A sequence that converges to 3 but is not monotonic.
   c) Two monotonic sequences with sum not monotonic.
3) Show that for all x, there exists a strictly increasing sequence of rational numbers converging to x. [Hint: Choose, be 'density of the rationals', $r_n \in (x - 2^{-n}, x - 2^{-(n+1)})$.]
4) Write .1247247247247... as a simple fraction.
5) Show that $\{r + \sqrt{2} : r \in \mathbf{Q}\} \cap \{r - \sqrt{2} : r \in \mathbf{Q}\}$ is empty.
6) Find $\sup\{x : 3x^2 + 3 < 10x\}$.
7) Find a number c such that $\dfrac{n}{n^2-1} \leq \dfrac{c}{n}$ for all $n \in \mathbf{N}$. [Hint: rewrite the inequality until you get an obvious choice for c, then write up a proof by going the other way.]
8) Show that $\{a + b\sqrt{2} : a, b \in \mathbf{Z}\}$ is countable. [Hint: Show the set is a countable union of countable sets; e.g., let $A_n := \{a + n\sqrt{2} : a \in \mathbf{Z}\}$ and define bijections $f_n : A_n \to \mathbf{Z}$ by $f_n(x) := x - n\sqrt{2}$.]
9) Find $\displaystyle\lim_{n\to\infty} \sum_{k=1}^{n} \frac{2^{k-1}}{(1-r)^{2k}}$ if $r < 1 - \sqrt{2}$. [Hint: geometric series].
10) Show that $(-1)^n$ is a divergent sequence.
11) Find the domains of $\dfrac{x^2}{1 - \sin x}$ and $\sqrt{x^2 + x - 5}$.
12) Show that if $x_n \to 3$, then $2x_n + 4 \to 10$.
13) Show that $\left(\dfrac{n+3}{6n-1}\right)$ is monotonic.
14) Show that $x := \sqrt{2} - \sqrt{3}$ is an algebraic number. [Hint: write out x and x^3 in terms of $\sqrt{2}$ and $\sqrt{3}$.]
15) Let $a_1 := 1$, $a_2 := 1$, and $a_{n+1} = \dfrac{a_n^2 + 1}{a_{n-1}}$
   a) Write out the first 6 terms of (a_n).
   b) Show, by induction, that $a_n \leq a_{n+1}$ for all n and therefore (a_n) is monotonic.
   c) Show that (a_n) diverges. [Hint: Suppose it has limit L. Use the defining equation for the sequence to show $L = (L^2 + 1)/L$ and thus get a contradiction].
16) Show that for all z, $\displaystyle\lim_{n\to\infty} \frac{\lfloor nz \rfloor}{n} = z$. [Hint: use the 'squeeze theorem'. Recall $\lfloor x \rfloor$ (the 'floor function' of x) is the largest integer less than or equal to x.]
17) Define $f(x) := x$ if x is rational, $f(x) := -x$ if x is irrational.
   a) Show that if x_n converges to 0, then $f(x_n)$ converges to $f(0)$.
   b) Show by example that if x_n converges to some $L \neq 0$, then $f(x_n)$ might not converge to $f(L)$.