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A note on color-transitivity of graphs.

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Given a graph X, an *n-coloring* is an assignment to each vertex one of n colors such that no two adjacent vertices are assigned the same color (i.e., a proper vertex n-coloring). An *admissible change of coloring* of an n-colored graph is the change of the color of one vertex so that the resulting graph is also n-colored. We say that two n-colorings are *equivalent* if there exists a sequence of admissible colorings changing one coloring to the other. This clearly defines an equivalence relation. Finally, we say that a graph X is *n-color transitive* if it has an n-coloring and all n-colorings are equivalent.

The term originates with Markov chain theory; randomly change the color of a random vertex with the condition that the result is an n-coloring. This defines a Markov chain and conditions on graphs have been studied where this chain is transient (i.e., where every state can be reached from every other). For example, if Δ denotes maximum vertex degree, then it is known that every graph is $(\Delta + 2)$-color transitive. See [1].

It turns out that every 3-color transitive graph is 2-colorable:

Theorem 1. *If a graph X is 3-color transitive then X is bipartite.*

Proof. Given a 3-coloring of X with colors chosen from $\{a, b, c\}$, orient each edge by the rule

$$a \to b, b \to c, c \to a.$$

Switching a and b of the given coloring gives a new 3-coloring with edge orientations exactly opposite and, assuming 3-color transitivity, the second coloring can be gotten to from the first by a sequence of admissible color changes. Note that any admissible change of color of a vertex changes the orientation of every edge incident with that vertex. Hence, going from the first coloring to the second, every edge

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changes orientation an odd number of times or, equivalently, if u and v are endpoints of an edge, then the number of times u changes plus the number of times v changes is odd. Therefore every edge connects a vertex that changes an even number of times with a vertex that changes an odd number of times. $\square$

The triangle is 3-colorable but not 3-color transitive. More generally, any cycle of length $3n$ is 3-colorable but not 3-color transitive. This shows that the converse of the theorem is false; the cycle of length 6 is bipartite but not 3-color transitive.

In general, the complete graph K_n is n-colorable but not n-color transitive. Also, any graph containing K_n as a sub-graph is not n-color transitive.

A natural generalization of the theorem is that if a graph is $(n+1)$-color transitive then X is n-colorable. We show that this is false for $n = 3$. Let X be the graph with adjacency matrix

$$\begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}$$

with respect to the basis $\{A, B, C, D, E, F, G\}$. (Two squares DFCA and DBEG with diagonals AF and BG respectively and an additional edge CE).

Theorem 2. *The graph X is 4-color transitive but not 3-colorable.*

Proof. First X is not 3-colorable. If it were, then either C, D are colored differently or D, E are and so either A, C, D, F require 4 colors or B, D, E, G do.

X is 4-colorable: let C, D be colored a, A, B be colored b, F, G be colored c, and E be colored d. We label this coloring $[a, b, c, d]$.

It is not hard to show that any coloring of X is equivalent to a coloring where A and B have the same color, C and D have the same color, and F and G have the same color (first make A and B agree, then F and G, and finally C and D). That is, every coloring by $\{a, b, c, d\}$ is equivalent to a coloring of the form $[\sigma(a), \sigma(b), \sigma(c), \sigma(d)]$ for some permutation σ of $\{a, b, c, d\}$.

If we let *'Zw'* denote the change of coloring of vertex Z to the color w, then it is easy to verify that the sequence *'Ad,Cb,Ea,Bd,Db'* takes $[a, b, c, d]$ to $[b, d, c, a]$ via admissible changes of coloring. Similarly, *'Ad,Fb,Ac,Dd,Ba,Gb,Bc,Da'* takes $[a, b, c, d]$ to $[a, c, b, d]$. It follows that every 4-coloring of X is equivalent to $[a, b, c, d]$. $\square$.

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[1] Frieze, Alan; Vigoda, Eric *A survey on the use of Markov chains to randomly sample colourings.* Combinatorics, complexity, and chance, 53–71, Oxford Lecture Ser. Math. Appl., 34, Oxford Univ. Press, Oxford, 2007.

Added in proof. The results here have been done independently of some earlier work by others; see L. Cereceda, J. van den Heuvel, and M. Johnson, *Connectedness of the graph of vertex-colourings*, Discrete Math. 308 (2008), 913-919.