$$Z(u) = \prod_{[c]} (1 - u^{\ell(c)})^{-1}$$

Euler product: $\prod_p (1 - p^{-s})^{-1} = \sum_n \frac{1}{n^s} = \zeta(s)$

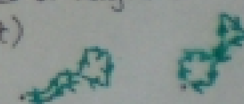
$$\prod_p \left(1 + \frac{1}{p-1}\right) = \prod_p \left(1 - \frac{1}{p}\right)^{-1} = \sum_n \frac{1}{n} = \infty$$

$$\sum_p \frac{1}{p-1} = \infty$$

A sum is associated to $Z(u)$ also:

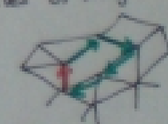
$$u\,[\log Z(u)]' = \sum_{n=1}^{\infty} b_n u^n$$

b_n = # non-backtracking tailless cycles of length n (with a designated starting point)



$$Z(u) = \exp\left(\sum_{n\geq 1} \frac{b_n}{n} u^n\right)$$

$a_n(e)$ = # non-backtracking tailless cycles of length n starting with e



$$Z_e(u) = \exp\left(\sum_{n\geq 1} \frac{a_n(e)}{n} u^n\right)$$

$$Z(u) = \prod_e Z_e(u)$$

$$Z(u) = \prod_{[c]} (1 - u^{\ell(c)})^{-1}$$

$$= \exp\left(\sum_{n\geq 1} \frac{b_n}{n} u^n\right)$$

$$= \det(L_u)$$

For G $(q+1)$-regular,

G is Ramanujan $\iff$ poles of $Z(q^{-s})$ satisfy $s = \frac{1}{2} + it$

"Riemann hypothesis for graphs"

For infinite graphs,

$$Z(u) = \exp\left(\sum_{n\geq 1}^{\infty} \frac{\tilde{b}_n}{n} u^n\right)$$

An alternative approach:

$$Z_e(u) := \exp\left(\sum_{n\geq 1}^{\infty} \frac{a_n(e)}{n} u^n\right)$$

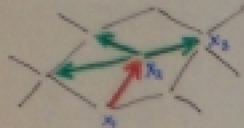
For many graphs:

$$Z(u) = \prod_e Z_e(u)$$

or

$$Z(u) = \prod_{e\in F} Z_e^{w(e)}(u)$$

Let M be the adjacency matrix of the 'directed edge graph' associated to G.

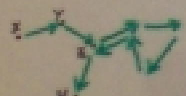


$M([x_1,x_2],[x_2,x_3]) = 1$

$$M([x,y],[z,w]) = \begin{cases} 1 & \text{if } z=y,\ w\sim y,\ w\neq x \\ 0 & \text{otherwise.} \end{cases}$$

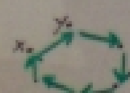
M is a $2\varepsilon \times 2\varepsilon$ matrix

$M^n([x,y],[z,w])$ = # non-backtracking paths $[x,y,\dots,z,w]$ of length n.



$n=7$

$M^n([x,y],[x,y])$ = # non-backtracking tailless cycles of length n starting at $[x,y]$.



$n=6$

$$\mathrm{tr}(M^n) = b_n$$

Let $L_u = \sum_{n=0}^{\infty} M^n u^n$; then $Z(u) = \det(L_u)$

$$\begin{aligned} u[\log Z(u)]' &= \sum_{n=1}^{\infty} b_n u^n \\ &= \sum_{n=1}^{\infty} \mathrm{tr}(M^n)u^n \\ &= \mathrm{tr}\Big(\sum_{n=1}^{\infty} M^n u^n\Big) \\ &= \mathrm{tr}(L_u - I) \end{aligned}$$

Jacobi: $(\log\det X)' = \mathrm{tr}(X'X^{-1})$

$$uL_u' = \sum_{n=1}^{\infty} nM^n u^n, \quad L_u^{-1} = I - uM, \quad uL_u'L_u^{-1} = L_u - I$$

$$\begin{aligned} u[\log\det L_u]' &= \mathrm{tr}(uL_u'L_u^{-1}) \\ &= \mathrm{tr}(L_u - I). \end{aligned}$$

$$Z(u) = \frac{1}{\det(I - uM)}$$

$Z(u)$ is a rational function

Another determinant formula:

$$Z(u) = \frac{\det(K_u)}{(1-u^2)^{\varepsilon}}$$

where K_u is the $\nu \times \nu$ matrix

$$K_u(x,y) = \sum_{n=0}^{\infty} \alpha_n(x,y)\, u^n$$

where $\alpha_n(x,y) = \#$ non-backtracking paths $x \to y$ (possibly with tails) of length n.

Since $(I - uA + u^2 Q) K_u = (1-u^2) I$,

$$Z(u) = \frac{1}{(1-u^2)^{\varepsilon-\nu} \det(I - uA + u^2 Q)}$$

Challenge: prove $\det(L_u) = \dfrac{\det(K_u)}{(1-u^2)^{\varepsilon}}$

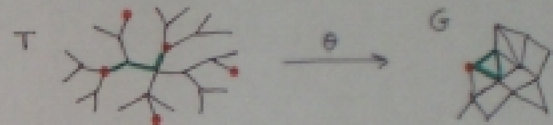
without using $(I - uA + u^2 Q) K_u = (1-u^2) I$

Purpose: possibly improves known proofs of 2^{nd} formula

$T :=$ universal covering tree

$\theta : T \to G$ preserves adjacency and vertex degree

(local homeomorphism)



$$K_u(x,y) = \sum_{\eta \in \theta^{-1}(y)} u^{d(\xi,\eta)} \quad \text{for (any) } \xi \in \theta^{-1}(x)$$

since $\alpha_n(x,y) = |S_n(\xi) \cap \theta^{-1}(y)|$ for any $\xi \in \theta^{-1}(x)$

and so $\sum_n \alpha_n(x,y) u^n = \sum_n u^n |S_n(\xi) \cap \theta^{-1}(y)| = \sum_n \sum_{\substack{\eta \in \theta^{-1}(y) \\ d(\xi,\eta) = n}} u^n$

$$L_u([x,y],[z,w]) = \sum_{(\mu,\nu) \in \theta^{-1}([z,w])} u^{\vec{d}([\xi,\eta],[\mu,\nu])} \quad \text{for (any) } [\xi,\eta] \in \theta^{-1}([x,y]).$$

G amenable $\overset{\text{def.}}{\Longleftrightarrow} \inf_S \dfrac{|\partial S|}{|S|} = 0$

$\Longleftrightarrow$ K_u and $K_u \mathbf{1}$ have same r.o.c.

[N - '02 ; requires 'quasi-regularity']

Note: the r.o.c. u_0 appears in 'Graph prime number theorem' : $n\pi(n) \sim u_0^{-n}$.

G $(q+1)$-regular. Ramanujan if $\lambda \in \operatorname{spec}(A)$

$\longrightarrow |\lambda| \leq 2\sqrt{q}$ or $\lambda = \pm(q+1)$.

Ramanujan $\Longleftrightarrow$ poles of $Z(q^{-s})$ satisfy $s = \frac{1}{2} + it$

"Riemann hypothesis for graphs".

let $\begin{cases} \psi(u) := \det(uI - M) \\ \phi(u) := \det(uI - A) \end{cases}$

$$\det(I - uM) = (1-u^2)^{\varepsilon - \nu} \det(I - uA + u^2 q I)$$

implies, using $\phi\left(\frac{1+qu^2}{u}\right) = \det\left(\frac{1+qu^2}{u} I - A\right) = \frac{1}{u^\nu} \det(I - uA + u^2 q I)$,

$$\psi(q^s) = q^{(2\varepsilon - \nu)s} (1 - q^{-2s})^{\varepsilon - \nu} \phi(q^s + q^{1-s})$$

s pole of $Z(q^{-s}) \Longleftrightarrow \psi(q^s) = 0$

$\Longleftrightarrow \phi(q^s + q^{1-s}) = 0$

$\Longleftrightarrow \begin{cases} s \in \mathbb{R} \longrightarrow q^s + q^{1-s} \geq 2\sqrt{q} & \text{(by Calculus)} \\ s \in \mathbb{C} \setminus \mathbb{R} \longrightarrow \operatorname{Re}(s) = \frac{1}{2} & (A \text{ symmetric} \rightarrow q^s + q^{1-s} \in \mathbb{R}) \end{cases}$

Problem: Definition and RH for non-regular graphs.

$\mathcal{N}_n([x,y]) := M^n([x,y],[x,y])$

$= \#$ non-backtracking, tailless cycles $\underbrace{[x,y,\ldots,x]}_{n}$

$\mathcal{N}_n(x) := \sum_{y \sim x} \mathcal{N}_n([x,y])$

$= \#$ non-backtracking, tailless cycles $\underbrace{[x,\ldots,x]}_{n}$

$b_n = \sum_x \mathcal{N}_n(x) = \sum_{[x,y]} \mathcal{N}_n([x,y])$ infinite for $|G| = \infty$

$$Z(u) = \exp\left(\sum_{n=1}^{\infty} \frac{b_n}{n} u^n\right)$$

$$= \prod_{[x,y]} \underbrace{\exp\left(\sum_{n=1}^{\infty} \frac{\mathcal{N}_n([x,y])}{n} u^n\right)}_{Z_{[x,y]}(u)} = \prod_x \underbrace{\exp\left(\sum_{n=1}^{\infty} \frac{\mathcal{N}_n(x)}{n} u^n\right)}_{Z_x(u)}$$

Grigorchuk & Żuk: replace b_n by an average $\tilde{b}_n$

$\tilde{b}_n = \lim_{k \to \infty} \frac{b_{n,X_k}}{|X_k|}$ and $Z(u) = \lim_{k \to \infty} Z_{X_k}(u)^{1/|X_k|}$

$\tilde{b}_n$ exists if $X_1 \leftarrow X_2 \leftarrow X_3 \leftarrow \ldots$ coverings

$\tilde{b}_n$ exists if $X_1 \subset X_2 \subset X_3 \ldots \nearrow G$ amenable

$\frac{|\partial X_n|}{|X_n|} \to 0$ $\qquad$ $\tilde{b}_n$ unique?

L_u exists for $u < u_0$; suppose $L_{u_0} = \infty$

Amenable $\Longrightarrow$ $L_u 1$ exists for $u < u_0$.

$\exists!\, h > 0$: $h = u_0 M h$, $h(e_0) = 1$

$$\lim_{u \nearrow u_0} \frac{L_u(e, e_0)}{L_u(e_0, e_0)} = h(e).$$

{ N. - 98 'Positive solutions of Schrödinger's equation on trees'

If $\lim\limits_{u \nearrow u_0} \dfrac{L_u f(e)}{L_u 1(e)}$ exists then it is independent of e

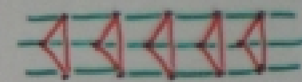
so provides a way to find an average of f.

$$Z_x(u) = \exp\left(\sum_{n \geq 1} \frac{\mathcal{B}_n(x)}{n} u^n\right)$$

For homogeneous graphs like $\mathbb{Z} \times \mathbb{Z}$, $\mathcal{B}_n(e)$ is the same for every e (and $\mathcal{B}_n(x)$ is the same for every x)

$$Z(u) = Z_x(u).$$

Ex: $\mathbb{Z} \times \mathbb{Z}_N$



two types of edges but $\mathcal{B}_n(x)$ same for every x :

$$Z(u) = Z_x(u)$$

In general, for Cayley graphs of a finitely generated group, every vertex 'looks the same' so $\mathcal{B}_n(x)$ constant

$$Z(u) = Z_x(u)$$

For periodic graphs (so only finitely many edge and vertex types),

$$\tilde{b}_n = \sum_{\substack{x \in S \\ \text{finite}}} \mathcal{B}_n(x)\, \omega(x)$$

$$Z(u) = \prod_{x \in S} Z_x^{\omega(x)}(u)$$

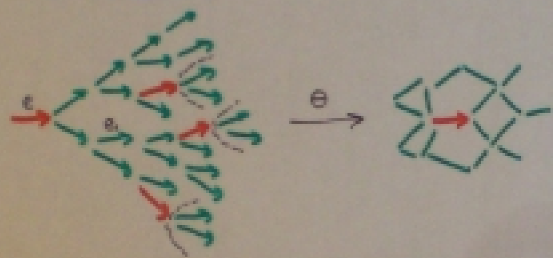
Is there a determinant formula for Z_e or Z_x ?

T = universal covering tree

$\theta : T \to G$

e an edge in T, $[e] := \theta^{-1}(\theta(e))$

$$M_e(e_1,e_2) = \begin{cases} 1 & \text{if } e_1 \to e_2 \text{ and } e_1 \notin [e] \\ 1 & \text{if } e \to e_2 \text{ and } e_1 \in [e] \\ 0 & \text{otherwise.} \end{cases}$$



$M_e^n(e_1,e_1)$ = # non-backtracking, tailless cycles of length n starting at $\theta(e_1)$ going through $\theta(e)$.

$$\mathrm{tr}(M_e^n) = \beta_n(e)$$

$$Z_e(u) = \det{}^{*}\Big(\sum_{n \geq 0} (uM_e)^n\Big) = \det{}^{*}(I - uM_e)^{-1}$$

Conclusions:

$$\begin{aligned} Z(u) &= \text{product} \\ &= \exp(\text{sum}) \\ &= \text{determinant} \end{aligned}$$

Determinant formula shows RH for graphs

For infinite graphs, definition of $Z(u)$ must be modified:

$$Z(u) = \exp\Big(\sum_{n=1}^{\infty} \frac{\tilde{b}_n}{n} u^n\Big)$$

For many cases, $\tilde{b}_n$ can be expressed in terms of $\beta_n(e)$ and so $Z(u)$ can be expressed in terms of

$$Z_e(u) = \exp\Big(\sum_{n=1}^{\infty} \frac{\beta_n(e)}{n} u^n\Big).$$

These 'zeta functions' exist for infinite graphs and have a determinant formula

$$Z_e(u) = \frac{1}{\det^{*}(I - uM_e)}$$

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