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QUASI-REGULAR GRAPHS, COGROWTH, AND AMENABILITY

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Abstract. We extend Grigrochuk's cogrowth criterion for amenability of groups to the case of non-regular graphs for which a certain regularity condition is satisfied. The proof involves generalized Laplacians which are inverses of growth series and whose determinants are closely related to zeta functions of graphs.

1. **Introduction.** The concept of amenability originated with von Neumann who once conjectured, though not in these words, that every non-amenable group is an extension of a free group on two generators. Roughly speaking, a group is amenable if the number of reduced words of length n grows at the same rate as the number of reduced words of length at most n. This conjecture was refuted only in 1984, by Ol'shanskii, who constructed a group which was neither a finite extension of F_2 nor was amenable. Showing that this group was not amenable utilized a "cogrowth criterion" for amenability first developed by Grigorchuk [5]. To understand this, note that every (finitely generated) group is the quotient of a free group. The cogrowth criterion for amenablility is that a group is amenable if and only if the size of spheres in the free group grow at the same rate as the intersections of those spheres with a fixed coset.

The concepts of amenability and cogrowth generalize easily to the graph setting. The author proved the cogrowth criterion for amenability for regular graphs [8]. Since concepts of amenability and cogrowth are geometric, it is conceivable that the cogrowth criterion extends to non-regular graphs as well. This is not true in full generality, however; see [10] for a counterexample. The aim of this paper is investigate sufficient conditions on a graph for the cogrowth criterion to hold.

To state our main result, we define "quasi-regularity". Given a graph G, a *path* is a sequence $x_0, x_1, \ldots, x_n$ of vertices in G such that x_{k-1} and x_k are adjacent for $k = 1, \ldots n$. A path is said to be *non-backtracking* if $x_k \neq x_{k+2}$ for all possible k. Given a finite non-backtracking path γ, let $\rho_{\gamma,n}$ be the number of non-backtracking paths which start with γ and go n further steps. A graph is said to be *quasi-regular* if the ratio $\rho_{\gamma_1,n}/\rho_{\gamma_2,n}$, as a function of γ_1, γ_2, and n is bounded. If s_n denotes the number of non-backtracking paths of length n and c_n denotes the number of non-backtracking paths which are actually cycles (i.e., for which $x_n = x_0$), then we may define growth and cogrowth constants:

$$gr(G) = \limsup_{n\to\infty} s_n^{1/n}, \qquad cogr(G) = \limsup_{n\to\infty} c_n^{1/n}. \tag{1}$$

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The main result of this paper is:

Theorem 1. *If G is quasi-regular then G is amenable if and only if $cogr(G) = gr(G)$.*

Our main interest in the cogrowth criterion is that, since it was so useful in the group setting, perhaps it is useful in the more general setting. In [10] , the cogrowth criterion was shown to hold for covers of finite graphs. Actually, it was shown to be true, more generally, when two analytical conditions hold [(A) and (B) in Section 3]. We shall present an improved proof of this fact in Section 3. In Section 4, we show that the purely geometric condition of quasi-regularity implies conditions (A) and (B).

2. **Basic Definitions.** Let G be a simple graph : i.e., a countable set (of vertices) equipped with a symmetric binary relation $\sim$ ($x \sim y$ means x and y share an edge and we say that x and y are adjacent). We define, for a vertex $x \in G$, $d(x)$ to be the number of vertices adjacent to x.

Following Gerl [4] , we say that G is amenable if

$$\inf_K \frac{|\partial K|}{|K|} = 0 \qquad (2)$$

where the infimum is over all finite non-empty subsets of G, and ∂K, the "boundary of K", is the set of edges between K and its complement.

The set $\mathbb{Z}^d$ can be thought of as a graph if we say two vertices are adjacent if and only if their ℓ^1 distance is 1. It is amenable since B_n, the metric ball of radius n, grows like n^d while S_n, the metric sphere of radius n, satisfies $|\partial B_n| \asymp |S_n| \asymp n^{d-1}$. The graph is then amenable.The graph $\mathbb{Z}^d$ is an example of a Cayley graph and, in fact, the Cayley graph of any finitely generated abelian group is amenable.

At the other extreme is the d-regular tree T_d ($d > 2$). If $K \subset T_d$, then the induced graph has a degree function d_K and it follows that

$$\frac{|\partial K|}{|K|} = \frac{1}{|K|} \sum_{x \in K} [d(x) - d_K(x)] = \frac{d|K| - 2|E_K|}{|K|} \geq d - 2$$

where E_K is the set of edges in K.

A *cover* of G is a graph H equipped with a local homeomorphism $f : H \to G$ (that is, f preserves vertex degree and adjacency). A maximal cover is necessarily a tree and, in fact, every graph has a unique (up to isomorphism) universal covering tree T. A way to think of T is to take the vertices of T to be the set of all finite non-backtracking paths starting at a fixed vertex where two paths are said to be adjacent if one is the extension of the other by a single step. We let θ denote the local homeomorphism $T \to G$. Fix a vertex $o \in T$. Given $\xi \in T$, let $|\xi|$ be the length of the shortest path from o to ξ. Let S_n denote the metric sphere of radius n (i.e., the set of vertices with $|\xi| = n$). Finally, we write, for real-valued functions A and B, $A \asymp B$ if A/B is bounded from both 0 and ∞. Then quasi-regularity can be redefined.

Definition 1. A graph is *quasi-regular* if

$$|S_{n+|\xi|} \cap T_z| \asymp |S_n|.$$

We may also reinterpret the cogrowth and growth constants in (1). Let $[o]$ be the coset $\{\xi : \theta(\xi) = \theta(o)\}$. Then

$$cogr(G) = \limsup_{n\to\infty} |S_n \cap [o]|^{\frac{1}{n}}$$

and

$$gr(G) = \limsup_{n\to\infty} |S_n|^{\frac{1}{n}}.$$

Let K_u be the "cogrowth series". That is, for $x, y \in G$, $\xi \in \theta^{-1}(x)$, and $\eta \in \theta^{-1}(y)$, let

$$K_u(x,y) = \sum_{n=0}^{\infty} |S_n(\xi) \cap [\eta]| u^n. \tag{3}$$

Since $K_u(x,y)$ can also be written as $\sum_{\rho\in[\eta]} u^{d(\xi,\rho)}$, it is clear that the convergence of K_u is independent of the choice of x and y. By (3), it is clear that K_u exists if $u < 1/cogr(G)$ but diverges if $u > 1/cogr(G)$. Even if K_u exists, $K_u f$ need not. Consider the "growth series"

$$K_u 1(x) = \sum_y K_u(x,y) = \sum_{n=0}^{\infty} |S_n(\xi)| u^n = \sum_\eta u^{d(\xi,\eta)}.$$

By the last equality, it is clear that the convergence of $K_u 1(x)$ is independent of x and, by the definition of $gr(T)$, $K_u 1$ exists if $u < 1/gr(T)$ and diverges if $u > 1/gr(T)$. For convenience, let

$$u_0 = 1/gr(T).$$

As a first step in studying the kernels K_u, we first find their inverses. A useful tool for this is the study of the "covering operators".

We say that a function $\hat{f} : T \to R$ covers $f : G \to R$ if

$$\hat{f} = f \circ \theta$$

and we say that a kernel (i.e., generalized matrix) $\widehat{M} : T \times T \to \mathbb{R}$ covers the kernel $M : G \times G \to \mathbb{R}$ if

$$M(\theta(\xi), \theta(\eta)) = \sum_{\rho\in[\eta]} \widehat{M}(\xi, \rho).$$

An example is given by the "adjacency matrices" of T and G: for $x, y \in G$, let $A(x,y)$ be 1 or 0 according to whether x and y are adjacent or not. Similarly, let $\widehat{A}$ denote the adjacency matrix of T. Since θ preserves vertex degree, $\widehat{A}$ covers A. Another such matrix is Q on G and $\widehat{Q}$ on T defined by

$$Qf(x) = (d(x) - 1)f(x).$$

Clearly, $\widehat{Q}$ covers Q.

It is easy to verify that the covering relation is preserved by matrix multiplication (i.e., $\widehat{MN} = \widehat{M}\widehat{N}$ by which we mean: if $\widehat{M}$ covers M and $\widehat{N}$ covers N, then $\widehat{M}\widehat{N}$ covers MN). Also, if $\widehat{f}$ covers f, then $\widehat{Mf} = \widehat{M}\widehat{f}$. If we define

$$\widehat{K}_u(x,y) = u^{d(x,y)},$$

then $\widehat{K}_u$ covers K_u as defined by (3).

Lemma 1. $(I - uA + u^2Q)K_u = K_u(I - uA + u^2Q) = (1 - u^2)I.$

Proof. Note that

$$\begin{aligned}\widehat{A}\widehat{K}_u(\xi,\eta) &= \sum_{\rho\sim\xi} u^{d(\rho,\eta)} \\ &= u^{d(\xi,\eta)}[(d(\xi)-1)u + 1/u - (1/u-u)\hat{I}(\xi,\eta)] \\ &= \widehat{K}_u(\xi,\eta)[d(\xi)u + (1/u-u)(1-\hat{I}(\xi,\eta))]\end{aligned}$$

and so $\widehat{A}\widehat{K}_u = u\widehat{D}\widehat{K}_u + (1/u-u)(\widehat{K}_u - \hat{I})$. Hence $AK_u = uDK_u + (1/u-u)(K_u - I)$ and so $(I - uA + u^2Q)K_u = (1-u^2)I$. The equality $K_u(I - uA + u^2Q) = (1-u^2)I$ can be treated similarly or by using the facts that $\widehat{K}_u\widehat{A}$ and $\widehat{K}_u\widehat{Q}$ are the transposes of $\widehat{A}\widehat{K}_u$ and $\widehat{Q}\widehat{K}_u$ respectively. □

We define, for $u \in \mathbb{R}$, a generalized *Laplacian* by $\Delta_u \equiv I - uA + u^2Q$. This terminology is motivated by the fact that $\Delta_1 = D - A$ is equivalent to the usual Laplacian on graphs $\Delta = D^{-1}A - I$. In general, Δ_u is equivalent to the Schrödinger operator $\Delta + q$ where $q(x) = u - u^2 - \frac{1-u^2}{d(x)}$. The operator Δ_u has long appeared (though not with this notation) in the literature on zeta functions for graphs. For example, Bass [1] was the first to prove:

$$Z(u) \equiv \prod_C (1 - u^{|C|})^{-1} = \frac{1}{(1-u^2)^r det(\Delta_u)}$$

where Z, the zeta function of a finite graph, is the product over "prime" cycles C, and r is the Betti number of the graph. See also the papers [6,9,11] for other proofs of this generalization of Ihara's theorem.

3. **Conditions for the cogrowth criterion.** Consider the following two conditions:

$$\exists h \asymp 1 : \Delta_{u_0} h = 0 \qquad (A)$$

and

$$K_u 1 \quad is \quad bounded \quad for \quad u < u_0. \qquad (B)$$

It turns out that these are sufficient for the cogrowth criterion to hold. This is a main result of the paper [10]. We present a simplified proof below.

Note that $cogr(G) < gr(G)$ is equivalent to the fact that the series $\sum |S_n|u^n$ and $\sum |S_n \cap [o]|u^n$ have different radii of convergence which is equivalent to the existence of $K_{u_0+\epsilon}$ for some $\epsilon > 0$.

We define the usual inner products on a graph's vertices and edges: if f_1 and f_2 are functions on the vertices of G, let

$$\langle f_1, f_2 \rangle = \sum_x f_1(x) f_2(x)$$

while if F_1 and F_2 are functions on directed edges, then

$$\langle F_1, F_2 \rangle = \frac{1}{2} \sum_{[x,y]} F_1(x,y) F_2(x,y).$$

A necessary and sufficient condition for $K_{u_0+\epsilon}$ to exist follows.

Proposition 1. *If G satisfies condition (B), then $K_{u_0+\epsilon}$ exists for some $\epsilon > 0$ if and only if $\Delta_{u_0} f \geq \lambda f$ for some positive function f and some $\lambda > 0$*

Proof. Suppose $f > 0$ and $\Delta_{u_0} f \geq \lambda f$ for some $\lambda > 0$. Since, for $\xi \in T$,

$$d(\xi) = |S_1(\xi)| \leq \frac{1}{u} \sum_{n \geq 0} |S_n(\xi)| u^n = \frac{1}{u} \widehat{K}_u 1(\xi),$$

there exists M such that $d(x) \leq M$ for all $x \in G$. Hence

$$Af(x) \leq (1 + u^2 q(x) - \lambda) f(x)/u \leq f(x)(1 + u^2(M-1) - \lambda)/u.$$

For $\sigma < \lambda$, we may then choose $\epsilon > 0$ such that $Af(x) \leq \frac{\sigma}{\epsilon} f$ and thus

$$-\epsilon A f + 2\epsilon u Q f + \epsilon^2 Q f \geq -\sigma f$$

and

$$\Delta_{u+\epsilon} f = f - uAf + u^2 Qf - \epsilon Af + 2\epsilon u Qf + \epsilon^2 Qf \geq (\lambda - \sigma) f.$$

Then $\widehat{\Delta}_{u_0+\epsilon} \hat{f} = \widehat{\Delta_{u_0} f} \geq (\lambda - \sigma)\hat{f}$. On T, $\widehat{K}_{u_0+\epsilon}$ exists and, by Lemma 1,

$$[1 - (u_0 + \epsilon)^2]\hat{f} = \widehat{K}_{u_0+\epsilon} \widehat{\Delta}_{u_0+\epsilon} \hat{f} \geq (\lambda - \sigma) \widehat{K}_{u_0+\epsilon} \hat{f}$$

from which it follows that

$$K_{u_0+\epsilon} f(x) \leq \frac{1 - (u_0 + \epsilon)^2}{\lambda - \sigma} f(x)$$

and therefore $K_{u_0+\epsilon}$ exists.

To prove the other way, suppose that, for some C,

$$\widehat{K}_{u_0} \widehat{K}_{u_0+\epsilon} \leq C \widehat{K}_{u_0+\epsilon}. \tag{4}$$

Then $g(x) \equiv K_{u_0+\epsilon}(x, x_0)$ satisfies $K_{u_0} g \leq Cg$ and $g \geq 0$. Choose λ such that $(1 - u_0^2) g \geq \lambda K_{u_0} g$. By Lemma 1, $\Delta_{u_0} K_{u_0} g \geq \lambda K_{u_0} g$. Letting $f = K_{u_0} g$, we find $f > 0$ and $\Delta_{u_0} f \geq \lambda f$.

It remains to prove (4) under the hypothesis that $K_u 1(x)$ is bounded for $u < u_0$. Fix $\xi, \eta \in T$ and suppose $\gamma = (\xi = \gamma_0, \gamma_1, \ldots, \gamma_n = \eta)$ is the path connecting ξ to η in T. Define $T(i) = \{\rho : d(\rho, \gamma_i) = d(\rho, \gamma)\}$. For convenience, let $s = u_0$ and $t = u_0 + \epsilon$. Then

$$\begin{aligned}
\widehat{K}_{u_0} \widehat{K}_{u_0+\epsilon}(\xi, \eta) &= \sum_{\rho} s^{d(\xi,\rho)} t^{d(\rho,\eta)} = \sum_{i=0}^{n} \sum_{\rho \in T(i)} s^{d(\rho,\gamma_i)+i} t^{d(\rho,\gamma_i)+n-i} \\
&= t^n \sum_{i=0}^{n} (s/t)^i \sum_{\rho \in T(i)} (st)^{d(\rho,\gamma_i)} = t^n \sum_{i=0}^{n} (s/t)^i \sum_{k=0}^{\infty} (st)^k |S_k(\gamma_i) \cap T(i)| \\
&\leq t^n \sum_{i=0}^{n} (s/t)^i \sum_{k=0}^{\infty} (st)^k |S_k(\gamma_i)| \leq t^n \frac{1}{1 - \frac{s}{t}} \sup_{\xi} \widehat{K}_{st} 1(\xi).
\end{aligned}$$

The result follows since $t^n = \widehat{K}_t(\xi, \eta)$ and $st < s = u_0$. □

It is well known [2,3] that the non-amenability of G is equivalent to

$$\inf_f \frac{\langle f, \Delta_1 f \rangle}{\langle f, f \rangle} > 0$$

where the infimum is over non-zero square summable functions on G. An equivalent condition is the existence of a positive solution to the inequality

$$\Delta_1 f \geq \lambda f$$

for some $\lambda > 0$. The equivalence of these last two conditions carries over to the operators Δ_u as well.

Proposition 2. *Suppose* $\exists h > 0 : \Delta_u h = 0$. *For* $\lambda \geq 0$, $\exists g > 0 : \Delta_u g \geq \lambda g$ *if and only if* $\inf_f \langle f, \Delta_u f\rangle / \langle f, f\rangle \geq \lambda$.

Proof. Suppose that $\Delta_{u_0} g \geq \lambda g$ for some $g > 0$. Define

$$\nabla_g f([x, y]) = \alpha(x, y)f(y) - \alpha(y, x)f(x)$$

where $\alpha(x, y) = \sqrt{u_0 g(x)/g(y)}$ and $[x, y]$ is a directed edge in G. Then, for square summable f_1, f_2,

$$\begin{aligned}\langle \nabla_g f_1, \nabla_g f_2\rangle &= \frac{1}{2}\sum_{[x,y]}[\alpha(x, y)f_1(y) - \alpha(y, x)f_1(x)][\alpha(x, y)f_2(y) - \alpha(y, x)f_2(x)] \\ &= \sum_x f_1(x)f_2(x)\sum_{y\sim x}\alpha(y, x)^2 - \sum_x f_1(x)\sum_{y\sim x}\alpha(x, y)\alpha(y, x)f_2(y).\end{aligned}$$

Since $\sum_{y\sim x}\alpha(y, x)^2 \leq 1 - u^2 + u^2 d(x) - \lambda$,

$$0 \leq \langle \nabla_g f, \nabla_g f\rangle \leq \langle f, \Delta_u f\rangle - \lambda\langle f, f\rangle$$

and therefore

$$\inf_f \langle f, \Delta_{u_0} f\rangle / \langle f, f\rangle \geq \lambda.$$

Let $K \subset G$ be finite and define Δ_K by

$$\Delta_K f = \Delta_{u_0}(\chi_K f)$$

and $\langle u, v\rangle_K = \sum_{x\in K} u(x)v(x)$. It is easy to see that

$$\langle f, \Delta_K g\rangle_K = \langle \chi_K f, \Delta_{u_0}(\chi_K g)\rangle_K$$

and thus Δ_K is self-adjoint and finite dimensional. Let $\mathcal{C}(K)$ be the functions supported on K and let

$$\lambda_K = \inf_{f\in\mathcal{C}(K)} \frac{\langle f, \Delta_K f\rangle_K}{\langle f, f\rangle_K}.$$

Suppose $\inf_f \langle f, \Delta_u f\rangle / \langle f, f\rangle \geq \lambda$. Then $\lambda_K \geq \lambda \geq 0$ and Δ_K is positive. Since K is finite, there exists an eigenvector f such that $\Delta_K f = \lambda_K f$. We argue that f can be assumed to be positive on K as follows. Let h be be positive and Δ_u-harmonic. Then

$$\begin{aligned}\langle f, \Delta_K f\rangle_K &= \langle \chi_K f, \Delta_{u_0}(\chi_K f)\rangle \\ &\geq \langle \nabla_h(\chi_K f), \nabla_h(\chi_K f)\rangle \\ &\geq \langle \nabla_h(\chi_K |f|), \nabla_h(\chi_K |f|)\rangle \\ &= \langle |f|, \Delta_K |f|\rangle_K\end{aligned}$$

where equality holds if and only if f does not change sign. However, since $\langle |f|, |f|\rangle = \langle f, f\rangle$ and $\langle |f|, \Delta_K |f|\rangle_K \geq \lambda_K$, equality indeed holds.

Let $o \in K_1 \subset K_2 \subset \ldots$ where $\cup K_i = G$ and define h_n to be a positive solution on K_n of $\Delta_{K_n} h_n \geq \lambda h_n$ normalized so that $h_n(o) = 1$. By Lemma 2.2, there exists a pointwise convergent subsequence and the pointwise limit, h, is positive and satisfies $\Delta_{u_0} h \geq \lambda h$. □

By combining propositions 1 and 2, we see that there is a gap between $cogr(G)$ and $gr(T)$ if and only if

$$\inf_f \frac{\langle f, \Delta_{u_0} f\rangle}{\langle f, f\rangle} > 0.$$

The cogrowth criterion then holds if this condition is equivalent to the non-amenability of G or, by [2],

$$\inf_f \frac{\langle f, \Delta_1 f\rangle}{\langle f, f\rangle} > 0.$$

With h as in (A), we indeed have this equivalence since, as in the proof of Proposition 2,

$$\begin{aligned}
\langle f, \Delta_{u_0} f\rangle &= \langle \nabla_h f, \nabla_h f\rangle \\
&= \frac{1}{2}\sum_{[x,y]}\Big(\sqrt{u_o \frac{h(x)}{h(y)}} f(y) - \sqrt{u_o \frac{h(y)}{h(x)}} f(x)\Big)^2 \\
&= \frac{u_0}{2}\sum_{[x,y]} h(x)h(y)\Big(\frac{f(x)}{h(x)} - \frac{f(y)}{h(y)}\Big)^2 \\
&\asymp \sum_{[x,y]}\Big(\frac{f(x)}{h(x)} - \frac{f(y)}{h(y)}\Big)^2 \\
&= \langle \frac{f}{h}, \Delta_1 \frac{f}{h}\rangle.
\end{aligned}$$

4. **Quasi-regularity.** Recall that a graph is quasi-regular if

$$S_{n+|\xi|} \cap T_\xi \asymp |S_n|.$$

Note that this is a property of the universal covering tree (as are properties (A) and (B)). Hence, if G is quasi-regular, then so is any cover of G and any graph covered by G.

Uniform trees (i.e., trees that cover a finite graph) have been studied by many authors. See for example [1]. A more general class of trees are those with finitely many cone types (in a sense, rooted trees which cover a directed graph). See [7], for example.

Proposition 3. *If T has finitely many cone types, then T is quasi-regular.*

Proof. Let $s_{n,\xi} = |S_{n+|\xi|} \cap T_\xi|$ for $\xi \in T$, and $n \geq 0$. Then

$$s_{n,\xi} = \sum_{\eta \in N(\xi)} s_{n-1,\eta}$$

where $N(\xi)$ is the set of all successors of vertices in T (i.e., the set of vertices η adjacent to ξ with $|\eta| = |\xi| + 1$).

We number the cone types and let the type of a vertex ξ be defined to be the type of the cone T_ξ. Consider the "type matrix" M defined by m_{ij} = the number of successors of a type i vertex which are of type j. Since $s_{n,\xi}$ is type-invariant, we may define $s_{n,i}$ in the obvious manner and we see that

$$s_{n,i} = \sum_j s_{n-1,j} m_{ij}.$$

Hence $s_n = Ms_{n-1}$ and therefore $s_n = M^n 1$. By the Perron-Frobenius theorem,

$$\frac{s_{n,i}}{s_{n,j}} \to \frac{v_i}{v_j}$$

where v is the Perron-Frobenius eigenvector of M (assuming M is primitive; a little more work must be done for the general case). □

We remark that conditions (A) and (B) also hold for covers of finite graphs. If G covers the finite graph G_0, then the lift $\hat{h}$ of a positive function h satisfying $\Delta_{u_0} h = 0$ on G_0 must satisfy condition (A) since h takes on only finitely many values. The existence of such a function on a finite graph follows from [6; Theorem 1.6]. Similarly, $K_u 1$ takes on only finitely many values when it exists and so it is bounded for all $u < u_0$. Hence condition (B) holds.

We now show that $quasi-regularity$ implies conditions (A) and (B). In the tree T with fixed vertex o, let Γ denote the set of all infinite non-backtracking paths starting at o. For $\xi \in T$, let Γ_ξ be the set of all paths in Γ which pass through the vertex ξ and define

$$T_\xi = \{\eta \in T : \exists \gamma \in \Gamma : \eta, \xi \in \gamma, |\eta| \geq |\xi|\} \cup \Gamma_\xi.$$

Such a set is called a "cone" and $\widehat{T} = T \cup \Gamma$, with the topology induced by the set of cones, is a compactification of T and its boundary, ∂T may be identified with Γ.

Every Borel measure ν (signed or not) on ∂T is totally determined by its values on cones. In particular, a set function ν on ∂T is a (signed) Borel measure if and only if for all $\xi \in T$,

$$\nu(T_\xi) = \sum_{\eta \in \xi^+} \nu(T_\eta) \tag{5}$$

where $\xi^+ = S_{|\xi|+1} \cap T_\xi$, the successors of ξ.

A source of such measures are $Delta_u$-harmonic functions. If $\Delta_u h = 0$, define $\nu(T_o) = (1-u^2)h(o)$ and, for $\xi \neq o$, $\nu(T_\xi) = u^{|\xi|}(h(\xi) - uh(\xi'))$ where ξ' predecessor of ξ. It is then easy to verify (5) for ν. The map $h \mapsto \nu$ is invertible and we find that

$$h(\xi) = u^{|\xi|}[\frac{\nu(T_o)}{1-u^2} + \sum_{k=1}^{|\xi|} u^{-2k} \nu(T_{\gamma_k})] \tag{6}$$

where $\gamma_0, \gamma_1, \ldots, \gamma_\xi$ is the non-backtracking path from o to ξ.

We remark that equation (6) is equivalent to a Martin representation for Δ_u-harmonic functions: for $\xi \in T$ and $\gamma \in \Gamma$, let $M_u(\xi, \gamma) = u^{2d(\xi,\gamma)-|\xi|}/(1-u^2)$. Then it is easy to rewrite (6) as

$$h(\xi) = \int_{\partial T} M_u(\xi, \gamma) \nu(d\gamma).$$

A difference in this case from the standard Martin representation theory is that it is possible that a positive harmonic function is represented by a non-positive (i.e. signed) measure ν.

For $u < u_0$, let

$$\mu_u(E) = \frac{\sum_{n \geq 0} |S_n \cap E| u^n}{\sum_{n \geq 0} |S_n| u^n}$$

and define h_u to be the function as defined by (6) where $\nu = \mu_u$. It is easy to see that

$$\sum_{\eta\in\xi^+} |S_n \cap T_\eta| = |S_n \cap T_\xi| - 1_{|\xi|}(n)$$

and therefore,

$$\sum_{\eta\in\xi^+} \mu_u(T_\eta) = \mu_u(T_\xi) - \frac{u^{|\xi|}}{K_u 1(0)}. \tag{7}$$

Theorem 2. *If G is quasi-regular, then conditions (A) and (B) hold.*

Proof. We first show that $K_{u_0}1 = \infty$. To see this, let $a_n = |S_n|$ and note that it is enough to show that $a_n \geq 1/Cu_0^n$ for all n. Since

$$a_{m+n} = \sum_{\xi\in S_m} |S_{m+n} \cap T_\xi| \leq C \sum_{\xi\in S_m} |S_n| = Ca_m a_n, \tag{8}$$

it is well known that $a_n^{\frac{1}{n}}$ converges and, by the definition of u_0 as the critical value of $\sum a_n u^n$,

$$a_n^{\frac{1}{n}} \to \frac{1}{u_0}.$$

By (8), it follows that

$$a_{nk} \leq C^{k-1} a_n^k$$

and thus, letting k go to infinity, $1/u_0 \leq C^{\frac{1}{n}} a_n^{\frac{1}{n}}$ and so $a_n \geq 1/Cu_0^n$ for all n.

We now show condition (A). Let $\mu(E) = \limsup_{n\to\infty} \mu_u(E)$. By (7) and the fact that $K_{u_0}1 = \infty$, μ is a measure and, since $\mu_u(T_\xi) \asymp u^{|\xi|}$, $\mu(T_\xi) \asymp u_0^{|\xi|}$. Let h be the Δ_{u_0}-harmonic function defined by (6) where $\nu = \mu$. Then, by (6), $h \asymp 1$.

We now show condition (B). Fix $\xi \neq o$ and define $s_n = |S_{n+|\xi|} \cap T_\xi|$ and $t_n = |S_{n+|\xi'|}| - |S_{n+|\xi'|} \cap T_{\xi'}|$. Then, for $n \geq 0$,

$$|S_n(\xi)| = s_n + t_{n-1}, |S_n(\xi')| = s_{n-1} + t_n.$$

If $S(u) = \sum s_n u^n$ and $T(u) = \sum t_n u^n$, then $K_u 1(\xi) = S(u) + uT(u)$, $K_u 1(\xi') = T(u) + uS(u)$ and therefore

$$K_u 1(\xi) - uK_u 1(\xi') = (1-u^2)S(u) = (1-u^2)K_u 1(o)\mu_u(T_\xi)u^{-|\xi|}.$$

It follows, by the definition of h_u, that

$$h_u(\xi) = \frac{K_u 1(\xi)}{(1-u^2)K_u 1(o)}.$$

By quasi-regularity, and the definition of μ_u, $\mu_u(T_\xi) \leq Cu^{|\xi|}$ for all ξ and thus, by (6), h_u is bounded. □

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